JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **136**, 404–418 (1988)

Namias' Fractional Fourier Transforms on L^2 and Applications to Differential Equations

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Submitted by Dr. R. P. Boas

Received March 3, 1987

INTRODUCTION

A recent paper [1] examined fractional powers of the Fourier transform on the Schwartz space $\mathscr{S}$. Its main purpose was to provide a rigorous framework in which the formal work of V. Namias [2] could be studied. This paper examines the theory on the space L^2 and gives some applications to partial differential equations.

1. PRELIMINARIES

The classical Fourier transform on $L^2 \equiv L^2(-\infty, \infty)$ is defined by

$$(\mathscr{F}f)(x) = \lim_{R \to \infty} \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{ixy} f(y)\, dy, \qquad f \in L^2, \tag{1.1}$$

where lim denotes the limit in mean square. For convenience we write

$$(\mathscr{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} f(y)\, dy, \qquad f \in L^p \tag{1.2}$$

although the integral must be interpreted in the sense of (1.1). We state the following (standard) theorem from [6, pp. 69–73].

THEOREM 1.1. *The operator* $\mathscr{F}$, *given by* (1.1), *is a homeomorphism on* L^2 *with inverse*

$$(\mathscr{F}^{-1}f)(x) = (\mathscr{F}f)(-x), \qquad f \in L^2.$$

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Further

$$\|\mathscr{F}f\|_2 = \|f\|_2 \qquad \forall f \in L^2.$$

We now recall from [1] the following definition.

DEFINITION 1.2. For suitable functions f, the fractional Fourier transform of order $\alpha \in \mathbb{R}$ is given by

$$\begin{aligned}
(\mathscr{F}_\alpha f)(x) &= \frac{e^{i((\pi/4)\hat{\alpha} - \alpha/2)}}{\sqrt{2\pi\,|\sin\alpha|}} \exp\left(-\frac{ix^2}{2}\cot\alpha\right) \\
&\quad \times \int_{-\infty}^{\infty} \exp\left(\frac{ixy}{\sin\alpha} - \frac{iy^2}{2}\cot\alpha\right) f(y)\,dy, \\
&\quad \hat{\alpha} = \operatorname{sgn}\alpha,\ 0 < |\alpha| < \pi, \qquad (1.3) \\
(\mathscr{F}_0 f)(x) &= f(x), \\
(\mathscr{F}_\pi f)(x) &= f(-x), \\
(\mathscr{F}_{\alpha+2\pi} f)(x) &= (\mathscr{F}_\alpha f)(x), \qquad \forall \alpha \in \mathbb{R}.
\end{aligned}$$

With this definition, we proved that for each $\alpha \in \mathbb{R}$, $\mathscr{F}_\alpha$ is a homeomorphism on $\mathscr{S}$ and

$$\mathscr{F}_\alpha \mathscr{F}_\beta f = \mathscr{F}_{\alpha+\beta} f, \qquad \forall \alpha, \beta \in \mathbb{R} \text{ and } \forall f \in \mathscr{S}. \tag{1.4}$$

The result (1.4) is known as the index law.

Notice now that when $n \in \mathbb{Z}$, $\mathscr{F}_{n\pi/2} f = \mathscr{F}^n f$ where $\mathscr{F}^n$ is the nth power of the classical Fourier operator (1.2). We therefore define the αth power of the Fourier transform by

$$\mathscr{F}^\alpha f = \mathscr{F}_{\alpha\pi/2} f.$$

Then we can deduce that

(i) $\mathscr{F}^\alpha : \mathscr{S} \to \mathscr{S}$ is a homeomorphism,

(ii) $\mathscr{F}^\alpha \mathscr{F}^\beta f = \mathscr{F}^{\alpha+\beta} f$, $\forall \alpha, \beta \in \mathbb{R}$ and $\forall f \in \mathscr{S}$.

Throughout this paper we shall be looking at properties of $\mathscr{F}_\alpha$. From these it is almost trivial to deduce results for $\mathscr{F}^\alpha$.

2. Fractional Fourier Transforms on L^2

When $f \in L^2$ we interpret (1.3) in the sense

$$(\mathscr{F}_\alpha f)(x) = A_\alpha \exp\left(-\frac{ix^2}{2}\cot\alpha\right)$$
$$\times \lim_{R\to\infty} \int_{-R}^{R} \exp\left(\frac{ixy}{\sin\alpha} - \frac{iy^2}{2}\cot\alpha\right) f(y)\, dy,$$

where lim denotes the limit in the mean and

$$A_\alpha = e^{i((\pi/4)\hat{\alpha} - \alpha/2)}/\sqrt{2\pi\,|\sin\alpha|}.$$

This coincides with (1.3) when $f \in \mathscr{S}$ and we can also write

$$(\mathscr{F}_\alpha f)(x) = A_\alpha \sqrt{2\pi} \exp\left(-\frac{ix^2}{2}\cot\alpha\right)$$
$$\times \mathscr{F}\left[\exp\left(-\frac{iy^2}{2}\cot\alpha\right) f(y)\right]\left(\frac{x}{\sin\alpha}\right), \qquad 0 < |\alpha| < \pi. \tag{2.1}$$

Therefore we can deduce

THEOREM 2.1. (i) *For each $\alpha \in \mathbb{R}$, $\mathscr{F}_\alpha$ is a homeomorphism on L^2 and* $\|\mathscr{F}_\alpha f\|_2 = \|f\|_2$, $\forall f \in L^2$.

(ii) *For each $\alpha \in \mathbb{R}$, and $f, g \in L^2$,*

$$\int_{-\infty}^{\infty} (\mathscr{F}_\alpha f)(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)(\mathscr{F}_\alpha g)(x)\, dx.$$

(iii) *For each $\alpha \in \mathbb{R}$, $\mathscr{F}_\alpha$ is a unitary operator on L^2.*

Proof. In each case, we can assume that $\alpha \in [-\pi, \pi]$ since the proof will then extend to $\alpha \in \mathbb{R}$ by periodicity.

(i) The cases $\alpha = 0$ and $\alpha = \pm\pi$ are trivial. When $0 < |\alpha| < \pi$, the result follows from (2.1) and Theorem 1.1.

(ii) Again the cases $\alpha = 0$ and $\alpha = \pm\pi$ are trivial. When $0 < |\alpha| < \pi$, the result follows from (2.1) and the Parseval relation

$$\int_{-\infty}^{\infty} (\mathscr{F} f)(x)\, g(x)\, dx = \int_{-\infty}^{\infty} f(x)(\mathscr{F} g)(x)\, dx, \qquad f, g \in L^2.$$

(iii) This follows from (ii) and the fact that for each $\alpha \in \mathbb{R}$, $\overline{\mathscr{F}_\alpha f} = \mathscr{F}_{-\alpha} \bar{f}$ where $\bar{f}(x)$ is the complex conjugate of $f(x)$, $\forall x \in \mathbb{R}$.

Let us now turn to the index law. We already know that (1.4) holds $\forall f \in \mathscr{S}$ and it is well known that $\mathscr{S}$ is dense in L^2. Therefore, from Theorem 2.1(i), we can extend (1.4) to $f \in L^2$ by arguments of continuity and density.

To summarise the results so far:

$$\mathscr{F}_\alpha : L^2 \to L^2 \text{ is a unitary operator } \forall \alpha \in \mathbb{R},$$

$$\mathscr{F}_\alpha \mathscr{F}_\beta f = \mathscr{F}_{\alpha+\beta} f, \qquad \forall \alpha, \beta \in \mathbb{R} \text{ and } \forall f \in L^2,$$

$$\mathscr{F}_0 f = f, \qquad \forall f \in L^2.$$

If we can also prove that

$$\|\mathscr{F}_\alpha f - f\|_2 \to 0 \qquad \text{as} \quad \alpha \to 0, f \in L^2, \tag{2.2}$$

then $\{\mathscr{F}_\alpha\}$ will be a strongly continuous unitary group of operators on L^2.

The following theorem shows that (2.2) is indeed true.

THEOREM 2.2. *For each* $f \in L^2$,

$$\|\mathscr{F}_\alpha f - f\|_2 \to 0 \qquad \textit{as} \quad \alpha \to 0.$$

Proof. Let $f \in L^2$ and let $\varepsilon > 0$ be arbitrary. Then, since $\mathscr{D}$ is dense in L^2, there exists $\psi \in \mathscr{D}$ such that $\|f - \psi\|_2 < \varepsilon/2$. Hence

$$\|\mathscr{F}_\alpha f - \mathscr{F}_\alpha \psi\|_2 = \|\mathscr{F}_\alpha (f - \psi)\|_2 = \|f - \psi\|_2 < \varepsilon/2, \qquad \forall \alpha \in \mathbb{R}.$$

We shall now prove that, for each $\varphi \in \mathscr{D}$,

$$\|\mathscr{F}_\alpha \varphi - \varphi\|_2 \to 0 \qquad \text{as } \alpha \to 0. \tag{2.3}$$

From this, we can deduce that

$$\begin{aligned}\|\mathscr{F}_\alpha f - f\|_2 &\leqslant \|\mathscr{F}_\alpha f - \mathscr{F}_\alpha \psi\|_2 + \|\mathscr{F}_\alpha \psi - \psi\|_2 + \|\psi - f\|_2 \\ &< \|\mathscr{F}_\alpha \psi - \psi\|_2 + \varepsilon\end{aligned}$$

so that, since ε was arbitrary,

$$\|\mathscr{F}_\alpha f - f\|_2 \to 0 \qquad \text{as } \alpha \to 0$$

as required.

To prove (2.3), suppose $\operatorname{supp} \varphi \subseteq [-R, R]$ for $0 < R < \infty$. We shall use the fact that $\mathscr{F} : L^2 \to L^2$ is a homeomorphism with $\|\mathscr{F}\|_2 = 1$. Hence proving (2.3) is equivalent to proving that

$$\|\mathscr{F}_{\alpha+\pi/2} \varphi - \mathscr{F} \varphi\|_2 \to 0 \qquad \text{as} \quad \alpha \to 0.$$

Let $\delta \in (0, \pi/4)$ be fixed and put

$$\begin{aligned}
H(\alpha) &= \|\mathscr{F}_{\alpha+\pi/2}\varphi - \mathscr{F}\varphi\|_2, \qquad 0 < |\alpha| < \delta < \frac{\pi}{4} \\
&= \Bigg\| \frac{1}{\sqrt{2\pi \cos\alpha}} e^{i(\pi/4 - \alpha/2 - \pi/4)} \int_{-\infty}^{\infty} \exp\left[\frac{i(x^2+y^2)}{2}\tan\alpha + \frac{ixy}{\cos\alpha}\right] \\
&\quad \times \varphi(y)\, dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy}\varphi(y)\, dy \Bigg\|_2 \\
&= \Bigg\| \sqrt{\frac{\cos\alpha}{2\pi}} e^{-i\alpha/2} \exp\left(\frac{ix^2}{2}\tan\alpha\right) \int_{-\infty}^{\infty} e^{ixy} \exp\left(\frac{iy^2}{2}\sin\alpha\cos\alpha\right) \\
&\quad \times \varphi(y\cos\alpha)\, dy - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy}\varphi(y)\, dy \Bigg\|_2 \\
&= \Bigg\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \Bigg[e^{-i\alpha/2}\sqrt{\cos\alpha} \exp\left(\frac{ix^2}{2}\tan\alpha + \frac{iy^2}{2}\sin\alpha\cos\alpha\right) \\
&\quad \times \varphi(y\cos\alpha) - \varphi(y) \Bigg] dy \Bigg\|_2 \\
&\leqslant \Bigg\| \sqrt{\frac{\cos\alpha}{2\pi}} e^{-i\alpha/2} \int_{-\infty}^{\infty} e^{ixy} \Bigg[\exp\left(\frac{ix^2}{2}\tan\alpha + \frac{iy^2}{2}\sin\alpha\cos\alpha\right) \\
&\quad \times \varphi(y\cos\alpha) - \varphi(y) \Bigg] dy \Bigg\|_2 \\
&\quad + \Bigg\| \frac{1}{\sqrt{2\pi}} [e^{-i\alpha/2}\sqrt{\cos\alpha} - 1] \int_{-\infty}^{\infty} e^{ixy}\varphi(y)\, dy \Bigg\|_2 \\
&\quad \text{(by the triangle inequality)} \\
&= \sqrt{\cos\alpha}\, \Bigg\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \Bigg[\exp\left(\frac{ix^2}{2}\tan\alpha + \frac{iy^2}{2}\sin\alpha\cos\alpha\right) \\
&\quad \times \varphi(y\cos\alpha) - \varphi(y) \Bigg] dy \Bigg\|_2 + |e^{-i\alpha/2}\sqrt{\cos\alpha} - 1|\, \|\mathscr{F}\varphi\|_2.
\end{aligned}$$

Now $|e^{-i\alpha/2}\sqrt{\cos\alpha} - 1|\, \|\mathscr{F}\varphi\|_2 \to 0$ as $\alpha \to 0$ and

$$\begin{aligned}
&\Bigg\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \left[\exp\left(\frac{ix^2}{2}\tan\alpha + \frac{iy^2}{2}\sin\alpha\cos\alpha\right) \varphi(y\cos\alpha) - \varphi(y) \right] dy \Bigg\|_2 \\
&\quad \leqslant \Bigg\| \frac{1}{\sqrt{2\pi}} \left[\exp\left(\frac{ix^2}{2}\tan\alpha\right) - 1 \right] \int_{-\infty}^{\infty} e^{ixy}\varphi(y)\, dy \Bigg\|_2 \\
&\qquad + \Bigg\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \left[\exp\left(\frac{iy^2}{2}\sin\alpha\cos\alpha\right) \varphi(y\cos\alpha) - \varphi(y) \right] dy \Bigg\|_2 .
\end{aligned}$$

Next,

$$\left\| \left[1 - \exp\left(\frac{ix^2}{2}\tan\alpha\right)\right](\mathscr{F}\varphi)(x)\right\|_2^2$$
$$= \int_{-\infty}^{\infty} \left| 1 - \exp\left(\frac{ix^2}{2}\tan\alpha\right)\right|^2 |(\mathscr{F}\varphi)(x)|^2\,dx \to 0 \qquad \text{as} \quad \alpha \to 0$$

by the dominated convergence theorem. Further,

$$\left\| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ixy} \left[\exp\left(\frac{iy^2}{2}\sin\alpha\cos\alpha\right)\varphi(y\cos\alpha) - \varphi(y)\right] dy \right\|_2$$
$$= \left\| \exp\left(\frac{iy^2}{2}\sin\alpha\cos\alpha\right)\varphi(y\cos\alpha) - \varphi(y)\right\|_2$$

by Parseval's equation

$$\leqslant \left\| \left[\exp\left(\frac{iy^2}{2}\sin\alpha\cos\alpha\right) - 1\right]\varphi(y)\right\|_2 + \|\varphi(y\cos\alpha) - \varphi(y)\|_2$$

and, as above,

$$\left\| \left[\exp\left(\frac{iy^2}{2}\sin\alpha\cos\alpha\right) - 1\right]\varphi(y)\right\|_2 \to 0 \qquad \text{as } \alpha \to 0$$

by the dominated convergence theorem.

Finally, we must consider

$$\|\varphi(y\cos\alpha) - \varphi(y)\|_2^2 = \int_{-\infty}^{\infty} |\varphi(y\cos\alpha) - \varphi(y)|^2\,dy.$$

Now $\operatorname{supp}\varphi \subseteq [-R, R]$ so $\varphi(y\cos\alpha) = 0$ whenever $|y\cos\alpha| \geqslant R$, i.e., $|y| \geqslant R/\cos\alpha$. Hence, when $0 < |\alpha| < \delta < \pi/4$, we have $|\varphi(y\cos\alpha) - \varphi(y)| = 0$ when $|y| > R/\cos\delta = T$. Therefore

$$\|\varphi(y\cos\alpha) - \varphi(y)\|_2^2 = \int_{-T}^{T} |\varphi(y\cos\alpha) - \varphi(y)|^2\,dy \to 0 \qquad \text{as } \alpha \to 0$$

by Lebesgue's theorem of bounded convergence. This completes the proof.

We can now state

THEOREM 2.3. *$\{\mathscr{F}_\alpha,\ \alpha \in \mathbb{R}\}$ is a strongly continuous unitary group of operators on L^2.*

We have to find the infinitesimal generator of $\{\mathscr{F}_\alpha\}$. This is, by definition, the operator $iG: L^2 \supseteq D(G) \to L^2$ given by

$$iGf = \lim_{\alpha \to 0} (1/\alpha)[\mathscr{F}_\alpha f - f],$$

$$D(G) = \left\{ f \in L^2 : \lim_{\alpha \to 0} (1/\alpha)[\mathscr{F}_\alpha f - f] \in L^2 \right\}.$$

Since $\{\mathscr{F}_\alpha\}$ is unitary, it follows that G is self-adjoint and is therefore closed. (See [5] for details.)

To find G, we shall need the following result from [2]:

$$\mathscr{F}_\alpha \left[\exp\left(-\frac{y^2}{2}\right) H_n(y) \right](x) = e^{in\alpha} \exp\left(-\frac{x^2}{2}\right) H_n(x), \qquad n = 0, 1, 2, ...,$$

where H_n is the nth degree Hermite polynomial defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

We shall show that $G = A$ where A is defined by Namias as

$$(Af)(x) = -\tfrac{1}{2} f''(x) + \tfrac{1}{2}(x^2 - 1) f(x)$$

and the domain of A is

$$D(A) = \{ f \in L^2 : f' \in L^2 \text{ and } f'' + (1 - x^2) f \in L^2 \}.$$

Note that $D(A)$ is dense in L^2 since it contains $\mathscr{D}$ and that A is self-adjoint. Let $f \in D(A)$. Then f can be represented as

$$f(x) = \sum_{n=0}^{\infty} a_n e^{-x^2/2} H_n(x),$$

where

$$a_n = \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) f(x)\, dx$$

and the series is convergent to $f(x)$ with respect to the norm in L^2. (See [7, p. 250] for details.) Hence, by the continuity of $\mathscr{F}_\alpha$,

$$\frac{1}{\alpha} [(\mathscr{F}_\alpha f)(x) - f(x)] = \sum_{n=0}^{\infty} \left(\frac{e^{in\alpha} - 1}{\alpha} \right) a_n e^{-x^2/2} H_n(x).$$

Further, since $Af \in L^2$ also, it can be represented as

$$(Af)(x) = \sum_{n=0}^{\infty} b_n e^{-x^2/2} H_n(x),$$

where the series converges in mean square to Af and

$$\begin{aligned} b_n &= \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} H_n(x)(Af)(x)\, dx \\ &= \frac{1}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} A[e^{-x^2/2} H_n(x)]\, f(x)\, dx \\ &\qquad \text{(performing the integrations by parts)} \\ &= \frac{n}{2^n n! \sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2/2} H_n(x)\, f(x)\, dx \\ &= n a_n. \end{aligned}$$

Note that we have used the result

$$A[e^{-x^2/2} H_n(x)] = n e^{-x^2/2} H_n(x)$$

which can be verified by differentiation. Therefore,

$$\begin{aligned} (iAf)(x) &= \sum_{n=0}^{\infty} i n a_n e^{-x^2/2} H_n(x) \\ &= \sum_{n=0}^{\infty} \lim_{\alpha \to 0} \left(\frac{e^{i\alpha n} - 1}{\alpha} \right) a_n e^{-x^2/2} H_n(x). \end{aligned} \tag{2.4}$$

We must now show that we can interchange the order of the limit and integration. To do this, we make use of the fact that $G\colon L^2 \supseteq D(G) \to L^2$ is closed.

It is obvious from (2.4) that when $f(x) = e^{-x^2/2} H_n(x)$, $\lim_{\alpha \to 0}(1/\alpha)[\mathscr{F}_\alpha f - f]$ exists in L^2 and equals iAf so that $\{\exp(-x^2/2)\, H_n(x)\}_{n=0}^{\infty} \subset D(G)$. Now define

$$f_m(x) = \sum_{n=0}^{m} a_n e^{-x^2/2} H_n(x)$$

so that f_m is the mth partial sum of $f \in D(A)$ and $\|f_m - f\|_2 \to 0$ as $m \to \infty$. Hence $f_m \in D(G)$ and

$$
\begin{aligned}
(iGf_m)(x) &= \lim_{\alpha \to 0} \sum_{n=0}^{m} \left(\frac{e^{i\alpha n} - 1}{\alpha} \right) a_n e^{-x^2/2} H_n(x) \\
&= \sum_{n=0}^{m} \lim_{\alpha \to 0} \left(\frac{e^{i\alpha n} - 1}{\alpha} \right) a_n e^{-x^2/2} H_n(x) \\
&= i \sum_{n=0}^{m} n a_n e^{-x^2/2} H_n(x);
\end{aligned}
$$

i.e., iGf_m is the mth partial sum of $iAf = -(i/2) f'' + (i/2)(x^2 - 1) f$ and so $\{Gf_m\}$ is convergent in the L^2 sense to Af. Finally, since G is closed, $f \in D(G)$ and

$$
Gf = \lim_{m \to \infty} Gf_m = -\tfrac{1}{2} f'' + \tfrac{1}{2}(x^2 - 1) f = Af.
$$

This holds for each $f \in D(A)$ so that $D(A) \subseteq D(G)$. Hence G is an extension of A. Therefore, since A and G are self-adjoint, it follows that $A = G$.

Consequently we can deduce

Result 2.4. When $g \in D(G)$, the (unique) classical solution to the problem

$$
\begin{aligned}
&\frac{\partial u}{\partial t} = -\frac{i}{2} \frac{\partial^2 u}{\partial x^2} + \frac{i}{2} (x^2 - 1) u, && x \in \mathbb{R},\ t \in \mathbb{R}, \\
&u(x, 0) = g(x), && x \in \mathbb{R},
\end{aligned} \tag{2.5}
$$

$u(x, t) \in L^2$ as a function of x for each fixed $t \in \mathbb{R}$,
is given by

$$
\begin{aligned}
u(x, t) = {} & \frac{e^{i((\pi/4)\hat{t} - t/2)}}{\sqrt{2\pi\, |\sin t|}} \exp\left(-\frac{ix^2}{2} \cot t \right) \\
& \times \int_{-\infty}^{\infty} \exp\left(\frac{ixy}{\sin t} - \frac{iy^2}{2} \cot t \right) g(y)\, dy, \qquad 0 < |t| < \pi,\ x \in \mathbb{R}, \\
& u(x, 0) = g(x), \qquad x \in \mathbb{R}, \\
& u(x, \pi) = g(-x), \qquad x \in \mathbb{R}, \\
& u(x, t + 2\pi) = u(x, t), \qquad t \in \mathbb{R},\ x \in \mathbb{R}.
\end{aligned}
$$

3. Applications to Differential Equations

In this section, we shall consider several variants of Result 2.4. In each case we shall solve the problem

$$\frac{\partial u}{\partial t} = iA_k u + f(t), \qquad t \in \mathbb{R}, \tag{3.1}$$

$$u(x, 0) = g(x), \qquad g \in D(A_k),$$

$$u(x, t) \in L^2(-\infty, \infty) \qquad \text{for each fixed } t \in \mathbb{R},$$

where $A_k: L^2(-\infty, \infty) \supseteq D(A_k) \to L^2(-\infty, \infty)$ is a specified operator which is independent of t and, for each fixed t, $(f(t))(x) = f_t(x) \in L^2(-\infty, \infty)$.

In Result 2.4, $f \equiv 0$ and

$$A = -\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}(x^2 - 1),$$

$$D(A) = \{f \in L^2(-\infty, \infty) : f'' \text{ exists and } f'' + (1 - x^2) f \in L^2(-\infty, \infty)\}.$$

EXAMPLE 3.1. *Let* $A_1 = \frac{1}{2}(d^2/dx^2) + \frac{1}{2}(1 - x^2) = -A$, $D(A_1) = D(A)$, $f \equiv 0$. Put $\tau = -t$ and $v(x, \tau) = u(x, t)$ in (3.1) to obtain

$$\frac{\partial}{\partial \tau} v(x, \tau) = iAv(x, \tau).$$

Hence

$$v(x, \tau) = (\mathscr{F}_\tau g)(x)$$

$$\Rightarrow u(x, t) = (\mathscr{F}_{-t} g)(x).$$

EXAMPLE 3.2. Let $f \equiv 0$ and define A_2: $L^2(-\infty, \infty) \supseteq D(A_2) \to L^2(-\infty, \infty)$ by

$$\begin{aligned} A_2 u &= \frac{1}{2}\frac{d^2 u}{dx^2} - \frac{1}{2} x^2 u \\ &= \frac{1}{2}\frac{d^2 u}{dx^2} + \frac{1}{2}(1 - x^2) u - \frac{1}{2} u \\ &= A_1 u + Bu, \qquad u \in D(A_2) = D(A), \end{aligned}$$

where $Bu = -\frac{1}{2}u$, $D(B) = L^2(-\infty, \infty)$. Therefore we need to solve

$$\frac{\partial u}{\partial t} = iA_1 u + iBu.$$

It is easy to see that $B\colon L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ is bounded and that $(iB)^n = (-i/2)^n I$, $\forall n \in \mathbb{N}$. Therefore

$$\sum_{n=0}^{\infty} \frac{(itB)^n}{n!} = e^{-it/2} I,$$

where I is the identity operator on $L^2(-\infty, \infty)$. Hence iB generates the C_0 group $\{S(t)\}$ where $(S(t)f)(x) = e^{-it/2} f(x)$, $f \in L^2(-\infty, \infty)$.

From above, we see that iA_1 generates the C_0 group $\{T(t)\}$ where $T(t) = \mathscr{F}_{-t}$. Next, it is easy to prove that $\forall s, t \in \mathbb{R}$ and $f \in L^2(-\infty, \infty)$,

$$S(s)\, T(t) f = T(t)\, S(s) f.$$

A standard theorem (see [4, p. 226]) now tells us that $iA_1 + iB$ generates the C_0 group $\{U(t)\}$ where $U(t) = S(t)\, T(t) = e^{-it/2} \mathscr{F}_{-t}$. Hence if $g \in D(A_2) = D(A)$, the unique L^2-solution of (3.1) with $k = 2$ is given by

$$u(x, t) = e^{-it/2} (\mathscr{F}_{-t}\, g)(x), \qquad x, t \in \mathbb{R}.$$

EXAMPLE 3.3. As above, $A_2 = \frac{1}{2}(d^2/dx^2 - x^2)$ and $D(A_2) = D(A)$ but this time $f \not\equiv 0$. Equation (3.1) now becomes

$$\frac{\partial u}{\partial t} = \frac{i}{2} \frac{\partial^2 u}{\partial x^2} - \frac{i}{2} x^2 u + f(x, t). \tag{3.2}$$

By variation of parameters, we obtain

$$u(x, t) = e^{-it/2} (\mathscr{F}_{-t}\, g)(x) + \int_0^t e^{-i(t-s)/2} (\mathscr{F}_{s-t} f)(x, s)\, ds, \tag{3.3}$$

where f is a suitably restricted function.

For example, when f is differentiable w.r.t. $t \in [0, T]$ for some $T > 0$ and

$$\int_0^T \left| \frac{\partial f}{\partial t}(x, t) \right| dt < \infty, \qquad \forall x \in \mathbb{R},$$

then (3.3) is the unique classical L^2-solution to (3.2) for $x \in \mathbb{R}$, $0 \leqslant t \leqslant T$. (For details, see [3, p. 109].)

EXAMPLE 3.4. This example is based on the following result of Romanoff [4]:

Let X and Y be Banach spaces with norms $\|\cdot\|_x$ and $\|\cdot\|_y$, respectively, and suppose $K\colon X \to Y$ is an isometry, i.e.,

$$\|Kf\|_y = \|f\|_x \qquad \forall f \in X.$$

Suppose also that A is the generator of a C_0 group $\{T(t)\}$ on X. Then $B = KAK^{-1}$ is the generator of the C_0 group $\{S(t)\}$ on Y where $S(t) = KT(t)K^{-1}$ and $D(B) = \{f \in Y\colon K^{-1}f \in D(A)\}$.

Let $X = L^2(-\infty, \infty)$ and $Y = \{f\colon \exp[\int_0^x \varphi(t)\,dt]\,f(x) \in L^2(-\infty, \infty)\}$ where $\varphi(x)$ is continuously differentiable on $\mathbb{R}$. Define

$$\|f\|_y = \left\| \exp\left[\int_0^x \varphi(t)\,dt\right] f(x) \right\|_2, \qquad f \in Y.$$

It is easy to check that $\|\cdot\|_y$ is a norm on Y. Therefore if we let

$$(Kf)(x) = \exp\left[-\int_0^x \varphi(t)\,dt\right] f(x), \qquad f \in X,$$

we can deduce that $K\colon X \to Y$ is an isometry with inverse

$$(K^{-1}f)(x) = \exp\left[\int_0^x \varphi(t)\,dt\right] f(x), \qquad f \in Y.$$

Next, take A to be the operator iA_1 where A_1 is given in Example 3.1 and $D(A_1) = D(A)$. Then, if $K^{-1}f \in D(A)$,

$$\begin{aligned}
&\left(K \frac{d^2}{dx^2} K^{-1}f\right)(x) \\
&\quad = \exp\left[-\int_0^x \varphi(t)\,dt\right] \frac{d^2}{dx^2} \left\{\exp\left[\int_0^x \varphi(t)\,dt\right] f(x)\right\} \\
&\quad = \exp\left[-\int_0^x \varphi(t)\,dt\right] \frac{d}{dx} \left\{\left[f'(x) + \varphi(x)\,f(x)\right] \exp\left[\int_0^x \varphi(t)\,dt\right]\right\} \\
&\quad = f''(x) + 2\varphi(x)\,f'(x) + [\varphi'(x) + \varphi^2(x)]\,f(x)
\end{aligned}$$

and

$$(K(1-x^2)\,K^{-1}f)(x) = (1-x^2)\,f(x).$$

Hence the unique solution of

$$\frac{\partial u}{\partial t} = \frac{i}{2}\left[\frac{\partial^2 u}{\partial x^2} + 2\varphi(x)\frac{\partial u}{\partial x} + (\varphi'(x) + \varphi^2(x) + 1 - x^2)\,u\right], \tag{3.4}$$

$$u(x, 0) = f(x)$$

in the class Y is given by

$$u(x, t) = (K\mathscr{F}_{-t}K^{-1}f)(x) \tag{3.5}$$

provided that $K^{-1}f \in D(A)$.

Calculating (3.5) explicitly, we obtain

$$\begin{aligned}(K\mathscr{F}_{-t}K^{-1}f)(x) &= A_{-t}\exp\left[\frac{ix^2}{2}\cot t - \int_0^x \varphi(y)\,dy\right] \\ &\qquad \times \int_{-\infty}^{\infty} \exp\left[\frac{iy^2}{2}\cot t - \frac{ixy}{\sin t}\right. \\ &\qquad \left. + \int_0^y \varphi(z)\,dz\right] f(y)\,dy, \qquad 0 < |t| < \pi, \\ (K\mathscr{F}_0K^{-1}f)(x) &= f(x), \\ (K\mathscr{F}_\pi K^{-1}f)(x) &= \exp\left[-\int_0^x \varphi(y)\,dy\right](K^{-1}f)(-x) \\ &= \exp\left[-\int_0^x \varphi(y)\,dy\right]\exp\left[\int_0^{-x} \varphi(y)\,dy\right] f(-x) \\ &= \begin{cases} \exp\left[-\int_{-x}^x \varphi(y)\,dy\right] f(-x), & x \geqslant 0 \\ \exp\left[\int_x^{-x} \varphi(y)\,dy\right] f(-x), & x < 0 \end{cases} \\ &= \exp\left[-\operatorname{sgn} x \int_{-|x|}^{|x|} \varphi(y)\,dy\right] f(-x)\end{aligned}$$

and $K\mathscr{F}_{-t}K^{-1}f$ is periodic in t with period 2π.

Particular cases of (3.4). (i) $\varphi(x) = i$:

$$\frac{\partial u}{\partial t} = \frac{i}{2}\left[\frac{\partial^2 u}{\partial x^2} + 2i\,\frac{\partial u}{\partial x} - x^2 u\right].$$

We have

$$\exp\left[-\int_0^x i\,dt\right] = e^{-ix}, \qquad Y = \{f\colon e^{ix}f(x) \in L^2(-\infty, \infty)\} = L^2(-\infty, \infty).$$

Further, it is easy to check that $Kf \in D(A)$ iff $f \in D(A)$. Hence, when $f \in D(A)$, the L^2-solution is

$$u(x, t) = A_{-t} \exp\left[\frac{ix^2}{2} \cot t - ix\right]$$

$$\times \int_{-\infty}^{\infty} \exp\left[-\frac{ixy}{\sin t} + \frac{iy^2}{2} \cot t + iy\right] f(y)\, dy,$$

$$x \in \mathbb{R},\ 0 < |t| < \pi,\ f \in D(A),$$

$$u(x, \pi) = e^{-2ix} f(-x), \qquad x \in \mathbb{R},$$

$$u(x, 0) = f(x), \qquad x \in \mathbb{R},$$

$$u(x, t + 2\pi) = u(x, t) \qquad \forall x, t \in \mathbb{R}.$$

(ii) $\varphi(x) = -x$:

$$\frac{\partial u}{\partial t} = \frac{i}{2}\left[\frac{\partial^2 u}{\partial x^2} - 2x \frac{\partial u}{\partial x}\right]$$

and

$$Y = \left\{f\colon \exp\left(-\frac{x^2}{2}\right) f(x) \in L^2(-\infty, \infty)\right\}.$$

The solution in the class Y is

$$u(x, t) = A_{-t} \exp\left(\frac{x^2}{2} + \frac{ix^2}{2} \cot t\right)$$

$$\times \int_{-\infty}^{\infty} \exp\left(-\frac{ixy}{\sin t} + \frac{iy^2}{2} \cot t - \frac{y^2}{2}\right) f(y)\, dy,$$

$$0 < |t| < \pi,\ x \in \mathbb{R},$$

$$u(x, \pi) = f(-x), \qquad x \in \mathbb{R},$$

$$u(x, 0) = f(x), \qquad x \in \mathbb{R},$$

$$u(x, t + 2\pi) = u(x, t), \qquad x, t \in \mathbb{R}$$

provided that $\exp(-x^2/2) f(x) \in D(A)$.

Note. In Example 3.4, we used similarity transforms to construct the unique solution to partial differential equations of type (3.4) subject to the given conditions. There are many other transformation methods available which enable us to solve more complicated problems. A very useful selection of these can be found in [4].

REFERENCES

1. A. C. McBRIDE AND F. H. KERR, On Namias' fractional Fourier transforms, *IMA J. Appl. Math.* **39** (1987), 159–175.
2. V. NAMIAS, The fractional order Fourier transform and its application to quantum mechanics, *J. Inst. Math. Its Appl.* **25** (1980), 241–265.
3. A. PAZY, "Semi-groups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, New York/Berlin, 1983.
4. N. P. ROMANOFF, On one-parameter groups of linear transformations, *Ann. of Math.* **48**, No. 2 (1947), 216–233.
5. M. H. STONE, On one-parameter unitary groups in Hilbert space, *Ann. of Math.* **33**, No. 2 (1932), 643–648.
6. E. C. TITCHMARSH, "An Introduction to the Theory of Fourier Integrals," Oxford Univ. Press, London/New York 1937.
7. A. H. ZEMANIAN, "Generalised Integral Transformations," Interscience, New York, 1968.